Weak and Strong Convergence Theorems for Normally Generalized Hybrid Mappings in Hilbert Spaces (Study on Nonlinear Analysis and Convex Analysis)

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Abstract. In this article, using Mann's type iteration, Halpern's type iteration, hybrid method and shrinking projection method, we obtain weak and strong convergence theorems for two generalized hybrid mappings and two normally 2-generalized hybrid mappings in a Hilbert space without assuming that they are commutative.

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1 Introduction

Let H be a real Hilbert space and let C be a nonempty subset of H. A mapping T from C into H is called *generic generalized hybrid* [21] if there exist $\alpha, \beta, \gamma, \delta \in \mathbb{R}$ such that $\alpha+\beta+\gamma+\delta \geq 0,\ \ \alpha+\beta>0$ and

$$\alpha\|Tx-Ty\|^2+\beta\|x-Ty\|^2+\gamma\|Tx-y\|^2+\delta\|x-y\|^2 \leq 0$$

for all $x, y \in C$. The class of generic generalized hybrid mappings covers generalized hybrid mappings defined by Kocourek, Takahashi and Yao [6]. A mapping $T: C \to H$ is called *generalized hybrid* [6] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Tx-Ty\|^2+(1-\alpha)\|x-Ty\|^2 \leq \beta\|Tx-y\|^2+(1-\beta)\|x-y\|^2$$

for all $x, y \in C$. The generalized hybrid mappings were extended by Maruyama, Takahashi and Yao [11] as follows: A mapping $T: C \to C$ is called 2-*generalized hybrid* [11] if there exist $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2\left\|T^2x-Ty\right\|^2&+\alpha_1\|Tx-Ty\|^2+(1-\alpha_1-\alpha_2)\|x-Ty\|^2\\ &\leq \beta_2\left\|T^2x-y\right\|^2+\beta_1\|Tx-y\|^2+(1-\beta_1-\beta_2)\|x-y\|^2\end{aligned}$$

for all $x, y \in C$. Very recently, 2-generalized hybrid mappings were extended by Kondo and Takahashi [7]. A mapping $T : C \to C$ is called *normally* 2-*generalized hybrid* [7] if there exist $\alpha_0, \beta_0, \alpha_1, \beta_1, \alpha_2, \beta_2 \in \mathbb{R}$ such that

$$\begin{aligned}\alpha_2\|T^2x - Ty\|^2 + \alpha_1\|Tx - Ty\|^2 + \alpha_0\|x - Ty\|^2 \\ + \beta_2\|T^2x - y\|^2 + \beta_1\|Tx - y\|^2 + \beta_0\|x - y\|^2 \leq 0\end{aligned}$$

for all $x, y \in C$, where $\sum_{n=0}^{2}(\alpha_n + \beta_n) \geq 0$ and $\alpha_2 + \alpha_1 + \alpha_0 > 0$.

In this article, using Mann's type iteration, Halpern's type iteration, hybrid method and shrinking projection method, we obtain weak and strong convergence theorems for two generalized hybrid mappings and two normally 2-generalized hybrid mappings in a Hilbert space without assuming that they are commutative.

2 Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We denote the strong convergence and the weak convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and $x_n \rightharpoonup x$, respectively. In a Hilbert space, it is known that

$$2\langle x - y, y\rangle \leq \|x\|^2 - \|y\|^2 \leq 2\langle x - y, x\rangle \tag{2.1}$$

for all $x, y \in H$ and

$$\|\alpha x + (1-\alpha)y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 - \alpha(1-\alpha)\|x - y\|^2 \tag{2.2}$$

for all $x, y \in H$ and $\alpha \in \mathbb{R}$; see [15]. Furthermore, we have that

$$2\langle x - y, z - w\rangle = \|x - w\|^2 + \|y - z\|^2 - \|x - z\|^2 - \|y - w\|^2 \tag{2.3}$$

for all $x, y, z, w \in H$. Let H be a Hilbert space and let C be a nonempty subset of H. Let T be a mapping of C into H. We denote by $A(T)$ the set of *attractive points* [17] of T, i.e.,

$$A(T) = \{z \in H : \|Tx - z\| \leq \|x - z\|, \quad \forall x \in C\}.$$

We also denote by $F(T)$ the set of fixed points of T. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called *quasi-nonexpansive* if

$$\|Tx - u\| \leq \|x - u\|, \quad \forall x \in C, \quad u \in F(T).$$

If C is closed and convex and $T : C \to H$ with $F(T) \neq \emptyset$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see Itoh and Takahashi [5]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_Dx\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive, i.e.,

$$\|P_Dx - P_Dy\|^2 \leq \langle P_Dx - P_Dy, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x \rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [14, 15]. Using this inequality and (2.3), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in D. \tag{2.4}$$

The following result was proved by Takahashi and Toyoda [19].

Lemma 2.1 ([19]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{x_n\}$ be a sequence in H. If $\|x_{n+1} - u\| \leq \|x_n - u\|$ for all $n \in \mathbb{N}$ and $u \in C$, then $\{P_C x_n\}$ converges strongly to $z \in C$, where P_C is the metric projection of H onto C.*

To prove one of our main results, we also need the following lemmas by Aoyama, Kimura, Takahashi and Toyoda [1, 24] and Maingé [9].

Lemma 2.2 ([1, 24]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence of $[0, 1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1 - \alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \ldots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.3 ([9]). *Let $\{X_n\}$ be a sequence of real numbers. Assume that $\{X_n\}$ is not monotone decreasing for sufficiently large $n \in \mathbb{N}$, in other words, there exists a subsequence $\{X_{n_i}\}$ of $\{X_n\}$ such that $X_{n_i} < X_{n_i+1}$ for all $i \in \mathbb{N}$. Let $n_0 \in \mathbb{N}$ such that $\{k \leq n_0 : X_k < X_{k+1}\} \neq \emptyset$. Define a sequence $\{\tau(n)\}_{n \geq n_0}$ of natural numbers as follows:*

$$\tau(n) = \max\{k \leq n : X_k < X_{k+1}\}, \quad \forall n \geq n_0.$$

Then, the followings hold:
(i) $\tau(n) \to \infty$ *as* $n \to \infty$;
(ii) $X_n \leq X_{\tau(n)+1}$ *and* $X_{\tau(n)} < X_{\tau(n)+1}, \quad \forall n \geq n_0.$

3 Weak convergence theorems of Mann's type iteration

In this section, using Lemma 2.1, we obtain a weak convergence theorem of Mann's type iteration [10] for finding a common attractive point of two generalized hybrid mappings without assuming that the mappings are commutative. Before proving the theorem, we need the following lemma.

Lemma 3.1. *Let H be a Hilbert space and let C be a nonempty subset of H. Let $T : C \to H$ be a generalized hybrid mapping and let $\{x_n\} \subset C$. If $x_n \rightharpoonup z$ and $x_n - Tx_n \to 0$, then $z \in A(T)$. Additionally, if C is closed and convex, then $z \in F(T)$.*

Theorem 3.2 ([16]). *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let S and T be generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Let $\{x_n\}$ be a sequence generated by $x_1 = x \in C$ and*

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)\big(\gamma_n S x_n + (1 - \gamma_n) T x_n\big), \quad \forall n \in \mathbb{N},$$

where $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$ *and* $\{\alpha_n\}$ *satisfy the following:*

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \alpha_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges weakly to a point* $z \in A(S) \cap A(T)$, *where* $z = \lim_{n\to\infty} P_{A(S)\cap A(T)} x_n$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges weakly to a point* $z \in F(S) \cap F(T)$, *where* $z = \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

We can also prove a weak convergence theorem by Mann's type iteration [10] for noncommutative two normally 2-generalized hybrid mappings in Hilbert spaces; see also [3].

Theorem 3.3 ([13])**.** *Let* H *be a Hilbert space and let* C *be a nonempty and convex subset of* H. *Let* S *and* T *be normally* 2*-generalized hybrid mappings of* C *into itself such that* $A(S) \cap A(T) \neq \emptyset$. *Given* $x_1 \in C$, *define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1} = a_n x_n + b_n\big(\gamma_n S + (1-\gamma_n)T\big)x_n + c_n\big(\delta_n S^2 + (1-\delta_n)T^2\big)x_n$$

for all $n \in \mathbb{N}$, *where* $a, b, c, d, e, f \in \mathbb{R}$ *and* $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ *satisfy the following:*

$$0 < a \leq \gamma_n \leq b < 1, \quad 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges weakly to a point* u *of* $A(S) \cap A(T)$, *where* $u = \lim_{n\to\infty} P_{A(S)\cap A(T)} x_n$.
Additionally, if C *is closed, then* $\{x_n\}$ *converges weakly to a point* $z \in F(S) \cap F(T)$, *where* $z = \lim_{n\to\infty} P_{F(S)\cap F(T)} x_n$.

4 Strong convergence theorems of Halpern's type iteration

In this section, using Lemmas 2.2 and 2.3, we prove the following strong convergence theorem of Halpern's type iteration [2] for noncommutative two generalized hybrid mappings in a Hilbert space; see also [22].

Theorem 4.1 ([16])**.** *Let* H *be a Hilbert space and let* C *be a nonempty and convex subset of* H. *Let* S *and* T *be generalized hybrid mappings of* C *into itself with* $A(S) \cap A(T) \neq \emptyset$. *Given* $x_1 \in C$ *and* $\{u_n\} \subset C$ *with* $u_n \to u$, *define a sequence* $\{x_n\}$ *in* C *as follows:*

$$x_{n+1} = \alpha_n u_n + (1-\alpha_n)\Big(\beta_n x_n + (1-\beta_n)\big(\gamma_n S x_n + (1-\gamma_n) T x_n\big)\Big)$$

for all $n \in \mathbb{N}$, *where* $a, b, c, d \in \mathbb{R}$, $\{\gamma_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *satisfy the following:*

$$\lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty,$$

$$0 < a \leq \gamma_n \leq b < 1 \quad \text{and} \quad 0 < c \leq \beta_n \leq d < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to* $P_{A(S)\cap A(T)} u$, *where* $P_{A(S)\cap A(T)}$ *is the metric projection from* H *onto* $A(S) \cap A(T)$. *Additionally, if* C *is closed, then* $\{x_n\}$ *converges strongly to* $P_{F(S)\cap F(T)} u$, *where* $P_{F(S)\cap F(T)}$ *is the metric projection from* H *onto* $F(S) \cap F(T)$.

We can also prove a strong convergence theorem by Halpern's type iteration [2, 23] for noncommutative two normally 2-generalized hybrid mappings in Hilbert spaces; see also [3, 8].

Theorem 4.2 ([13]). *Let H be a Hilbert space and let C be a nonempty and convex subset of H. Let S and T be normally 2-generalized hybrid mappings of C into itself such that $A(S) \cap A(T) \neq \emptyset$. Given $x_1, z \in C$, define a sequence $\{x_n\}$ in C as follows:*

$$\begin{cases} x_{n+1} = \lambda_n z + (1-\lambda_n) z_n, \\ z_n = a_n x_n + b_n\big(\gamma_n S + (1-\gamma_n)T\big)x_n + c_n\big(\delta_n S^2 + (1-\delta_n)T^2\big)x_n, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c, d, e, f \in \mathbb{R}$ and $\{\lambda_n\}, \{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$ satisfy the following:

$$\lim_{n\to\infty} \lambda_n = 0, \quad \sum_{n=1}^{\infty} \lambda_n = \infty,$$
$$0 < a \leq \gamma_n \leq b < 1, \quad 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to $z_0 = P_{A(S)\cap A(T)} z$, where $P_{A(S)\cap A(T)}$ is the metric projection from H onto $A(S) \cap A(T)$.

Additionally, if C is closed, then $\{x_n\}$ converges strongly to $P_{F(S)\cap F(T)} z$, where $P_{F(S)\cap F(T)}$ is the metric projection from H onto $F(S) \cap F(T)$.

5 Strong convergence theorems by hybrid methods

In this section, we obtain a strong convergence theorem by the hybrid method [12] for finding a common fixed point of two generalized hybrid mappings without assuming that the mappings are commutative.

Theorem 5.1 ([4]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\Big(\gamma_n S x_n + (1-\gamma_n) T x_n\Big), \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $a, b, c \in \mathbb{R}$ and $\{\alpha_n\}, \{\gamma_n\} \subset [0,1]$ satisfy

$$0 \leq \alpha_n \leq a < 1 \quad \text{and} \quad 0 < b \leq \gamma_n \leq c < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Next, we prove a strong convergence theorem by the shrinking projection method [18] for noncommutative two generalized hybrid mappings in a Hilbert space.

Theorem 5.2 ([4]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)\big(\gamma_n S x_n + (1-\gamma_n) T x_n\big), \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ is the metric projection of H onto C_{n+1} and $b, c \in \mathbb{R}$ and $\{\alpha_n\}, \{\gamma_n\} \subset [0,1]$ satisfy

$$0 \leq \liminf_{n\to\infty} \alpha_n < 1 \quad \text{and} \quad 0 < b \leq \gamma_n \leq c < 1, \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} u$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Furthermore, using the hybrid method [12], we prove a strong convergence theorem for noncommutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 5.3 ([20]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\gamma_n S + (1-\gamma_n)T\big)x_n + c_n\big(\delta_n S^2 + (1-\delta_n)T^2\big)x_n, \\ C_n = \{z \in C : \|y_n - z\| \leq \|x_n - z\|\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_n\cap Q_n}$ is the metric projection of H onto $C_n \cap Q_n$ and $a, b, c, d, e, f \in \mathbb{R}$ and $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0,1]$ satisfy the following:

$$0 < a \leq \gamma_n \leq b < 1, \quad 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to $z_0 = P_{F(S)\cap F(T)} x_1$, where $P_{F(S)\cap F(T)}$ is the metric projection of H onto $F(S) \cap F(T)$.

Finally, we prove a strong convergence theorem by the shrinking projection method [18] for noncommutative normally 2-generalized hybrid mappings in a Hilbert space.

Theorem 5.4 ([20]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $S, T : C \to C$ be normally 2-generalized hybrid mappings such that $F(S) \cap F(T) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Let $C_1 = C$ and let $\{x_n\} \subset C$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases} y_n = a_n x_n + b_n\big(\gamma_n S + (1-\gamma_n)T\big)x_n + c_n\big(\delta_n S^2 + (1-\delta_n)T^2\big)x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} u_{n+1}, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $P_{C_{n+1}}$ *is the metric projection of* H *onto* C_{n+1} *and* $a, b, c, d, e, f \in \mathbb{R}$ *and* $\{\gamma_n\}, \{\delta_n\}, \{a_n\}, \{b_n\}, \{c_n\} \subset [0, 1]$ *satisfy the following:*

$$0 < a \leq \gamma_n \leq b < 1, \quad 0 < c \leq \delta_n \leq d < 1,$$
$$a_n + b_n + c_n = 1 \quad \text{and} \quad 0 < e \leq a_n, b_n, c_n \leq f < 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ *converges strongly to* $z_0 = P_{F(S)\cap F(T)}u$*, where* $P_{F(S)\cap F(T)}$ *is the metric projection of* H *onto* $F(S) \cap F(T)$*.*

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